

Localization of Eigenstates in One-Dimensional Disordered Systems

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Exact results are obtained on the localization of eigenstates in one-dimensional infinite disordered systems with diagonal and off-diagonal randomnesses.

A Furstenberg-type theorem is established for the product of matrices associated with a multi-Markov-chain.

As a result, Matsuda and Ishii's theory is generalized to examine the systems with both randomnesses. Harmonic chains, tightly binding electronic systems and Heisenberg-Mattis model are considered as typical examples.

§ 1. Introduction

From the pioneering work of Anderson¹⁾ problem of localization of one-particle wave functions, caused by some randomnesses, has become one of the difficult but attractive problems in many fields of physics.

Historically many attempts have been made to understand the feature of localization. Numerical studies on finite systems brought us many striking features of the localized states.^{2),3)} Theoretical studies, however, often failed to reproduce the features and brought us to the conclusions which were not true.

Economou, Cohen⁴⁾ and Licciardello⁵⁾ are good successors of Anderson's approach. They have proposed some general criterions on the localization of eigenstates. The $F(E)$ method or $L(E)$ method is applicable to many problems without the restrictions of the dimension and the structure of the system. However, the accuracies of their results are still obscure.

Scaling and renormalization group are useful methods to study the problem.^{6)~9)} They are expected to give us a sharp criterion whether states are localized or not. Some of the theoretical bases are, however, intuitive ones and justification of this approach seems yet to be left.

We should remark the other two. The one is the recent progress on perturbative approach^{10)~12)} and the other are some discussions on one-dimensional conductors.^{13)~16)}

Concerning one-dimensional systems we can find some theories which have little obscurity, such as Herbert and Jones¹⁷⁾ and Thouless's¹⁸⁾ relation. Especially Matsuda and Ishii's probabilistic approach (referred to as MI¹⁹⁾ and I²⁰⁾) on the exponential growth of each particular wave function is a precious

one which gives us meaningful rigorous conclusions. In general probabilistic treatments bring us much correct information on random systems and this is the approach through which we want to discuss the localization problem.

The purpose of this paper is thus to extend their theory to be applicable to general one-dimensional systems with off-diagonal randomness. It becomes then necessary to generalize Furstenberg's convergent theorem²¹⁾ to the case of a product of matrices associated with a multi-Markov-chain. The Furstenberg-type theorem thus obtained plays an essential role in this paper. A part of the generalization has been given in its original form,²²⁾ so attention is paid here to further generalization of the theory and also to some applications. As is noted in § 2 we can discuss a wide variety of systems with the theory. We need not necessarily restrict our discussion to the system with nearest neighbour interaction. However, rather simple examples are discussed. One-dimensional Heisenberg-Mattis model with random exchange interactions and random expectation values of S^z , harmonic chain with mass and force constant randomnnesses and one-dimensional tightly binding electronic system with diagonal and off-diagonal randomnnesses, are good simple examples to see the applicability of the theory.

In the next section model systems which can be treated rather easily are described. In § 3 general aspects of the theory are shown using these examples. In § 4 a Furstenberg-type theorem is derived in a general form. The theorem is applied to our models in § 5. Conclusions and discussions are given in the final section.

§ 2. Model systems

The systems which we are interested in are one-dimensional infinite linear systems with short-range interaction. Theoretical restrictions to the system which come from the conditions in the Furstenberg-type theorem will be seen in § 4 on the following points.

- a) The ensemble of systems can be considered as a stationary (multi-) Markov-process.
- b) $\lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\prod_{i=1}^n |\det T_i| \right) = 0, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\prod_{i=-n+1}^0 |\det T_i| \right) = 0.$
- c) $Q^{*(i)} \in SL(m, R). \quad (i = 1 \sim s)$

These are quite natural restrictions and under which a wide variety of systems exist. The theory presented in this paper is thus applicable to these systems under the restrictions. The application of the theory is, however, limited in this paper to some typical model systems written below which can be treated rather easily.

Model 1

Harmonic chain The model is represented by an equation of motion

$$m_n \frac{d^2 u_n}{dt^2} = t_{n,n+1} u_{n+1} - (t_{n,n+1} + t_{n,n-1}) u_n + t_{n,n-1} u_{n-1},$$

$$(-\infty \leq n \leq \infty) \quad (2.1)$$

where

$$t_{n,n+1} = t_{n+1,n} \quad \text{and} \quad 0 < t_{n,n+1}, m_n < \infty. \quad (2.2)$$

There are three typical cases.

1a) Masses m_n take, mutually independently, r' different values with a common probability distribution,

$$P'(m_n = m^j) = P'^j \quad (\text{independent of } n), \quad j=1, \dots, r',$$

$$\sum_{j=1}^{r'} P'^j = 1. \quad (2.3)$$

The force constants are described as

$$t_{n,n+1} = t(m_n, m_{n+1}) = t(m_{n+1}, m_n). \quad (2.4)$$

When $t(m_n, m_{n+1}) = \text{constant}$, only diagonal randomness (DR) exists in the system.

1b) Force constants $t_{n,n+1}$ take, mutually independently, r different values with a common probability distribution,

$$P(t_{n,n+1} = t^i) = P^i \quad (\text{independent of } n), \quad i=1, \dots, r,$$

$$\sum_{i=1}^r P^i = 1. \quad (2.5)$$

The masses are described as

$$m_n = m(t_{n,n+1}, t_{n,n-1}) = m(t_{n,n-1}, t_{n,n+1}). \quad (2.6)$$

When $m_n = \text{constant}$, only the randomness of force constant exists in the system. Case 1b) is equivalent to case 1a) through Toda's dual transformation.²⁴⁾

1c) m_n take, mutually independently, r' different values with a common distribution and at the same time $t_{n,n+1}$ also take, mutually independently, r different values with a common distribution,

$$P'(m_n = m^j) = P'^j, \quad (\text{independent of } n), \quad j=1, \dots, r',$$

$$\sum_{j=1}^{r'} P'^j = 1,$$

$$P(t_{n,n+1} = t^i) = P^i, \quad (\text{independent of } n), \quad i=1, \dots, r,$$

$$\sum_{i=1}^r P^i = 1. \quad (2.7)$$

It should be remembered here that Eq. (2.1) is rewritten as,

$$\omega^2 u_n' = t'_{n,n+1} u_{n+1}' + (-t'_{n,n+1} - t'_{n,n-1}) u_n' + t'_{n,n-1} u_{n-1}', \quad (2.1')$$

where

$$t'_{n,i} = \frac{-t_{n,i}}{\sqrt{m_n m_i}} \quad \text{and} \quad u_n' = \sqrt{m_n} u_n. \quad (2.1'')$$

Model 2

Tightly binding electronic system The model is described by the Hamiltonian

$$H = \sum_n |n\rangle \varepsilon_n \langle n| + \sum_n (|n\rangle t_{n,n+1} \langle n+1| + |n+1\rangle t_{n+1,n} \langle n|), \quad (-\infty \leq n \leq \infty) \quad (2.8)$$

where

$$t_{n,n+1} = t_{n+1,n} \quad \text{and} \quad 0 < t_{n,n+1} < \infty. \quad (\text{for all } n) \quad (2.9)$$

In this model DR and off-diagonal randomness (ODR) means the randomnesses of atomic energies ε_n and transfer integrals $t_{n,n+1}$ respectively.

It should be mentioned that the assumption (2.9) does not restrict the problem because any randomness of phase of transfer integrals $t_{n,n+1}$ can be eliminated by a transformation in which only the phases of the basic functions change.²⁵⁾ Thus a system with complex transfer integrals is equivalent to a system with no phase randomness.

The following three typical cases are considered.

2a) Like case 1a), atomic energies ε_n take, mutually independently, r' different values with a common distribution. When $t_{n,n+1} = t(\varepsilon_n, \varepsilon_{n+1}) = \text{constant}$ we have systems with DR only. It should be noted that excitonic systems of Frenkel-type can also be considered to be in this case.

2b) Like case 1b), the transfer integrals take, mutually independently, r different values with a common distribution. When $\varepsilon_n = \varepsilon(t_{n,n+1}, t_{n,n-1}) = \text{constant}$ we have systems with ODR only.

2c) Like case 1c), the atomic energies take, mutually independently, r' different values with a common distribution and at the same time the transfer integrals also take, mutually independently, r different values with a common distribution.

Model 3

*Heisenberg-Mattis model*²⁶⁾ As an simple example of magnetic system the following Heisenberg-Mattis model is considered:

$$H = -\frac{1}{2} \sum_n \xi_n \xi_{n+1} J_{n,n+1} S_n \cdot S_{n+1} - h \sum_n \xi_n S_n^z, \quad (2.10)$$

$$J_{n,n+1} = J_{n+1,n} > 0, \quad \xi_n = \pm 1, \quad (\text{for all } n) \quad (2.11)$$

where $\xi_n \xi_{n+1} J_{n,n+1}$ and S_n for all n are exchange interactions and spin operators, respectively. The quantities ξ_n (for all n) are random variables which take the values ± 1 and the values of the quantities $\xi_n \xi_{n+1}$ ($= \pm 1$) determine the sign of the exchange interaction. For example, when $\xi_n = 1$ (or -1) for all n we have ferromagnetic systems and when $\xi_n = -\xi_{n+1} = 1$ (or -1) for all n we have anti-ferromagnetic systems. In one-dimensional case with nearest-neighbour interaction, any disordered system with ferromagnetic and anti-ferromagnetic exchange interactions can be described by the Heisenberg-Mattis model.

The linearized equation of motion for $S^+ = \{S_n^+\}$ about the ground state $|0\rangle$ is²⁰⁾

$$i \frac{\partial}{\partial t} S^+ = \mathcal{O}(h) U S^+, \quad (2.12)$$

where

$$U_{n,n'} = \xi_n \delta_{n,n'}, \quad (2.13)$$

$$\mathcal{O}(h)_{n,n'} = \left(\sum_k J_{n,k} S_k + h \right) \delta_{n,n'} - J_{n,n'} S_n, \quad (2.14)$$

$$\xi_n S_n \equiv \langle 0 | S_n^z | 0 \rangle. \quad (S_n > 0) \quad (2.15)$$

An equivalent equation is derived for S^- and the corresponding equation can be derived from (2.10) formally by using the following Holstein-Primakoff-type transformation,²⁰⁾

$$\begin{aligned} S_n^z &= \xi_n (S_n - a_n^+ a_n), \\ S_n^+ &\simeq (2S_n)^{1/2} \left\{ \frac{(1+\xi_n) a_n}{2} + \frac{(1-\xi_n) a_n^+}{2} \right\}, \\ S_n^- &\simeq (2S_n)^{1/2} \left\{ \frac{(1+\xi_n) a_n^+}{2} + \frac{(1-\xi_n) a_n}{2} \right\}. \end{aligned} \quad (2.16)$$

The eigenvalue equation for magnons is derived from (2.12) as

$$S^{-1} \omega S^+ = \mathcal{O}^* U S^+, \quad \mathcal{O}^* = S^{-1} \mathcal{O}, \quad (2.17)$$

$$(US)^{-1} \omega \phi = \mathcal{O}^* \phi, \quad \phi = U S^+, \quad (2.18)$$

which is written in our case as

$$t_{n,n} \phi_n + t_{n,n+1} \phi_{n+1} + t_{n,n-1} \phi_{n-1} = \frac{E}{\xi_n S_n} \phi_n,$$

$$t_{n,n} = \frac{S_{n+1}}{S_n} J_{n,n+1} + \frac{S_{n-1}}{S_n} J_{n,n-1} + \frac{\hbar}{S_n},$$

$$t_{n,n\pm 1} = -J_{n,n\pm 1}. \quad (2.19)$$

It is almost impossible to know the ground state $|0\rangle$ of each disordered system and hence we cannot know the values $\{\xi_n S_n\}$ for each system. We thus assume, like models 1 and 2, that ξ_n , S_n and $J_{n,n\pm 1}$ for all n are random variables written in the following ten cases.

3a₀) $S_n = S$ for all n and only ξ_n 's take mutually independently two different values ± 1 with a common distribution. This represents the original Heisenberg-Mattis model when $J_{n,n+1} = J$ for all n . In our case $J_{n,n+1} = J(\xi_n, \xi_{n+1})$.

3a₁) $\xi_n = 1$ (or -1) for all n and only S_n 's take mutually independently r' different values with a common distribution. This represents ferromagnetic random systems. $J_{n,n+1} = J(S_n, S_{n+1})$.

3a₂) $\xi_{2n} = -\xi_{2n+1} = 1$ (or -1) for all n and only S_n 's take mutually independently r' different values with a common distribution. This represents anti-ferromagnetic random systems. $J_{n,n+1} = J(S_n, S_{n+1})$.

3a₃) ξ_n 's take mutually independently two values ± 1 with a common distribution and at the same time S_n 's take mutually independently r' different values with a common distribution. $J_{n,n+1} = J(\xi_n, S_n, \xi_{n+1}, S_{n+1})$.

3b₁) $\xi_n = 1$ (or -1) and $S_n = S > 0$ for all n and only $J_{n,n+1}$'s take mutually independently r'' different values with a common distribution.

3b₂) $\xi_{2n} = -\xi_{2n+1} = 1$ (or -1) and $S_n = S > 0$ for all n and only $J_{n,n+1}$'s take mutually independently r'' different values with a common distribution.

3b₃) $S_n = S > 0$ for all n and ξ_n 's take mutually independently two values ± 1 with a common distribution and at the same time $J_{n,n+1}$'s take mutually independently r'' different values with a common distribution.

3c₁) $\xi_n = 1$ (or -1) for all n and S_n 's and $J_{n,n+1}$'s take, respectively, mutually independently r' and r'' different values with common distributions.

3c₂) $\xi_{2n} = -\xi_{2n+1} = 1$ (or -1) for all n and S_n 's and $J_{n,n+1}$'s take, respectively, mutually independently r' and r'' different values with common distributions.

3c₃) ξ_n 's take mutually independently two values ± 1 , with a common distribution, and at the same time S_n 's and $J_{n,n+1}$'s take, respectively r' and r'' different values with common distributions.

A question arises whether $\Phi(h)U > 0$ for $h \rightarrow 0^+$ or not. It may be true in the ferromagnetic cases (3a₁, 3b₁ and 3c₁) but it is not guaranteed in general. We proceed further in this paper leaving the question.

§ 3. Exponential growth of particular solution

When we put $\bar{\alpha}_n$ for each model

$$\bar{\alpha}_n = \begin{cases} (t_{n,n+1} + t_{n,n-1}) - m_n \omega^2, & \text{(model 1)} \\ (\epsilon - \varepsilon_n), & \text{(model 2)} \\ \left(\frac{\epsilon}{\xi_n S_n} - t_{n,n} \right), & \text{(model 3)} \end{cases} \quad (3.1)$$

the eigenvalue equation is written as

$$\begin{pmatrix} \phi_{n+1} \\ \phi_n \end{pmatrix} = \begin{pmatrix} \bar{\alpha}_n/t_{n,n+1}, & -t_{n,n-1}/t_{n,n+1} \\ 1, & 0 \end{pmatrix} \begin{pmatrix} \phi_n \\ \phi_{n-1} \end{pmatrix} = T_n \begin{pmatrix} \phi_n \\ \phi_{n-1} \end{pmatrix}, \quad (3.2)$$

where

$$\begin{aligned} T_n &= \sqrt{\det T_n} \begin{pmatrix} \frac{\bar{\alpha}_n \text{sign}(t_{n,n+1})}{\sqrt{t_{n,n+1} t_{n,n-1}}}, & -\sqrt{\frac{t_{n,n-1}}{t_{n,n+1}}} \\ \sqrt{\frac{t_{n,n+1}}{t_{n,n-1}}}, & 0 \end{pmatrix} \\ &= \sqrt{\det T_n} \cdot Q_n(t_{n,n+1}, \bar{\alpha}_n, t_{n,n-1}). \end{aligned} \quad (3.3)$$

Obviously $\det Q_n = 1$. The following relation holds about the exponential growth of the norm of X_n for any $X_0 \in (R^m - \{0\})^*$ where

$$\begin{aligned} \|X_n\| &\equiv \phi_n^2 + \phi_{n+1}^2 + \dots + \phi_{n+m-1}^2, \\ X_n &= \left(\prod_{i=1}^n T_i \right) X_0, \end{aligned} \quad (3.4)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \| \left(\prod_{i=1}^n T_i \right) X_0 \| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \| \left(\prod_{i=1}^n Q_i \right) X_0 \|. \quad (3.5)$$

Our main purpose is to discuss whether the limit (3.5) exists and the value of which is finite and positive, i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \| \left(\prod_{i=1}^n Q_i \right) X_0 \| = 2\gamma > 0 \quad (3.6)$$

or not, independent of $X_0 \in (R^m - \{0\})$ and of sample systems. For this purpose it is necessary to generalize Furstenberg's convergent theorem to the case of a product of matrices representing a multi-Markov-chain. This exponential growth property of particular solution plays an essential role in the localization problem. Once the exponential growth property (3.6) is guaran-

*) Integer m is the dimension of the transfer matrix and $m-1$ usually means the range of the interaction of the system. Though $m=2$ in our simple models and our examinations are made only on these models, property (3.6) will be related with a Furstenberg-type theorem in §4 for general m .

***) For general m , Q is defined as $Q = |\det T|^{-1/m} T$ and it is supposed that $\lim_{n \rightarrow \infty} (1/n) \log \prod_{i=1}^n |\det T_i| = 0$.

teed, many conclusions are obtained using some other related theorems which has essentially been given in the previous paper.

§ 4. A Furstenberg-type theorem

4.1) Let \mathcal{Q}^0 be a set of all sample systems of the type discussed in § 2, including that of each simple model. A physically reasonable measure μ_1^0 can be introduced on \mathcal{Q}^0 in essentially the same way as that in I; μ_1^0 can be extended to a complete measure on the whole Borel sets of the interval $\Sigma[0, 1]$. It will be seen later that it becomes more convenient to omit from \mathcal{Q}^0 a set of special samples $\{\omega_j^0\}$ with a sufficiently small measure $\varepsilon_N > 0$ in order to avoid a mathematical difficulty which occurs when we apply Furstenberg's convergent theorem. It will be seen that we can make the measure $\varepsilon_N = \mu^0(\{\omega_j^0\}) > 0$ as small as we hope by making the set $\{\omega_j^0\}$ as special one as we hope, so that the subtraction does not affect physical phenomena. We thus define $\mathcal{Q} = \mathcal{Q}^0 - \{\omega_j^0\}$. Obviously a complete measure $\mu = \mu^0 / (1 - \varepsilon_N)$ is meaningful (as $\mu^0(\{\sigma_j^0\}) = \varepsilon_N$) also on the smallest Borel sets including the intervals $\Sigma[0, 1] - \{\sigma_j^0\}$, where $\{\sigma_j^0\}$ is a set of intervals corresponding to the set $\{\omega_j^0\}$. We will use the expression $\mu^0(\{\omega_j^0\})$ for $\mu^0(\{\sigma_j^0\})$ in this paper.

Let

$$(\dots, a_{-1}, a_0, a_1, \dots) \tag{4.1}$$

be a stationary (multi-) Markov-process representing \mathcal{Q}^0 . Random variable a_n can be, for example, $m_n, t_{n,n+1}$ or m_n and $t_{n,n+1}$ in cases 1a), 1b) and 1c) respectively. a_n thus take, in general, r different real values

$$a^j \quad (j=1 \sim r, r < \infty) \tag{4.2}$$

independently of n . For any $\omega \in \mathcal{Q}^0$

$$(\dots, a_{-1}(\omega), a_0(\omega), a_1(\omega), \dots) \tag{4.3}$$

represents ω completely. Corresponding to ω or the representation of ω a sequence of the transfer matrices

$$(\dots, T_{-1}(\omega), T_0(\omega), T_1(\omega), \dots) \tag{4.4}$$

or that of more suitably defined Q 's

$$(\dots, Q_{-1}(\omega), Q_0(\omega), Q_1(\omega), \dots) \tag{4.4'}$$

can be defined. Let $Q_n(\omega)$ or $T_n(\omega)$ can be determined for each n by $n' + n'' + 2$ a 's

$$(a_{n-n'}(\omega), a_{n-n'+1}(\omega), \dots, a_{n+n''}(\omega), a_{n+n''+1}(\omega)). \tag{4.5}$$

When we put

$$\alpha_n = (a_{n-n'}, a_{n-n'+1}, \dots, a_{n+n'}, a_{n+n'+1}), \quad (4.6)$$

that is,

$$\begin{aligned} \alpha_n(\omega) = & (a_{n-n'}(\omega), a_{n-n'+1}(\omega), \dots \\ & \dots, a_{n+n'}(\omega), a_{n+n'+1}(\omega)), \end{aligned} \quad (4.6')$$

$Q_n(\omega)$ can be determined by $\alpha_n(\omega)$. α_n takes s different values independently of n

$$\alpha^i, \quad (i=1 \sim s, s \leq r^{n'+n'+2}) \quad (4.7)$$

The quantities $\{\alpha^i\}$ thus describe a local character of samples around a site, for example, n .

The set of sample systems \mathcal{Q} can be decomposed with the $\{\alpha^i\}$, in one way, as follows:

$$\mathcal{Q} = \sum_{i=1}^s \mathcal{Q}^{(i)},$$

$$\mathcal{Q}^{(i)} = \{\omega; \alpha_0(\omega) = \alpha^i \text{ for any } \omega \in \mathcal{Q}^{(i)}\}; \quad (i=1 \sim s) \quad (4.8)$$

$\mathcal{Q}^{(i)}$ is a subset of \mathcal{Q} composed of systems for which

$$\alpha_0(\omega) = \alpha^i. \quad (4.9)$$

When we write

$$\begin{aligned} \mu(\mathcal{Q}^{(i)}) = \nu^{(i)} = \mu^0(\mathcal{Q}^{(i)}) / (1 - \epsilon_N) = P^{(i)}(1 - \epsilon_N^{(i)}) / (1 - \epsilon_N), \\ (i=1 \sim s) \end{aligned} \quad (4.10)$$

each $(\mathcal{Q}^{(i)}, B^{(i)}, \mu^{(i)} = \mu/\nu^{(i)})$ becomes a probability space.

4.2) On each sample $\omega \in \mathcal{Q}^{(i)}$ ($i=1 \sim s$), the l -th "irreducible sequence" of the i -th kind $S^{(i)}$, and the corresponding l -th "irreducible transfer matrix" of the i -th kind $Q^{*(i)}$, are defined as follows: An "irreducible sequence" of the i -th kind $S^{(i)}$ is a sequence of $\alpha(\omega)$'s

$$(\alpha_{n+1}(\omega), \alpha_{n+2}(\omega), \dots, \alpha_{n+n'}(\omega)) \quad (4.11)$$

which fulfill the conditions that

(1) $\alpha_n(\omega) = \alpha^{(i)}$, (2) $\alpha_{n+n'}(\omega) = \alpha^{(i)}$ and (3) no other $\alpha(\omega)$'s in the sequence are equal to $\alpha^{(i)}$. An "irreducible transfer matrix" of the i -th kind $Q^{*(i)}$ is a product of transfer matrices Q corresponding to the "irreducible sequence" $S^{(i)}$.

The irreducible sequences and transfer matrices of the i -th kind are intro-

duced in order to describe the right semi-infinite chain of $\omega \in \mathcal{Q}^{(i)}$ which start from $\alpha_0(\omega) = \alpha^{(i)}$. Obviously the left semi-infinite chain of $\omega \in \mathcal{Q}^{(i)}$ can be described with the correspondingly defined sequences and matrices. More precisely, a right semi-infinite chain of $\omega \in \mathcal{Q}^{(i)}$ can be represented by an infinite sequence of the irreducible sequences

$$(S_1^{(i)}, S_2^{(i)}, \dots, S_l^{(i)}, \dots) \tag{4.12}$$

with probability 1, and also by a product of the irreducible transfer matrices

$$(\dots \cdot Q_l^{*(i)} \cdot \dots \cdot Q_2^{*(i)} \cdot Q_1^{*(i)}) \tag{4.13}$$

with probability 1. As a simple example the irreducible sequences and the irreducible transfer matrices of model 1b) or 2b) are shown in Table I, for the case $r=2$ and $i=1$, with their values of probability distribution $\mu^{(i)}(Q^{*(i)}) = \mu^{(i)}(S^{(i)})$.

Table I. An example of the irreducible sequences, the corresponding irreducible transfer matrices and the values of the distribution of model 1b) or 2b) in the case $r=2$ and $i=1$. It is shown in §4 that $\epsilon_N^{(1)}$ is equal to the sum of $(1-P^{(1)})^l P^{(1)}$ ($l=N^{(1)} \sim \infty$) in this case and thus equal to $(1-P^{(1)})/P^{(1)}$.

$S^{(1)}$	$Q^{*(1)}(S^{(1)})$	$\mu^{(1)}(Q^{*(1)}) = \mu^{(1)}(I(S^{(1)}))$
$(\epsilon^{(1)}) \setminus \epsilon^{(1)}$	$Q(\epsilon^{(1)}, \epsilon^{(1)})$	$P^{(1)}/(1-\epsilon_N^{(1)})$
$(\epsilon^{(1)}) \setminus \epsilon^{(1)} \epsilon^{(1)}$	$Q(\epsilon^{(1)}, \epsilon^{(1)}) Q(\epsilon^{(1)}, \epsilon^{(1)})$	$(1-P^{(1)}) \cdot P^{(1)}/(1-\epsilon_N^{(1)})$
\vdots	\vdots	\vdots
$\overset{n-1}{\epsilon^{(1)} \setminus \overbrace{\epsilon^{(1)} \epsilon^{(1)} \dots \epsilon^{(1)} \epsilon^{(1)}}}$	$Q(\epsilon^{(1)}, \epsilon^{(1)}) (Q(\epsilon^{(1)}, \epsilon^{(1)}))^{(n-1)} Q(\epsilon^{(1)}, \epsilon^{(1)})$	$(1-P^{(1)})^{(n-1)} \cdot P^{(1)}/(1-\epsilon_N^{(1)})$
\vdots	\vdots	\vdots
$\overset{N^{(1)}-1}{\epsilon^{(1)} \setminus \overbrace{\epsilon^{(1)} \epsilon^{(1)} \dots \epsilon^{(1)} \epsilon^{(1)}}}$	$Q(\epsilon^{(1)}, \epsilon^{(1)}) (Q(\epsilon^{(1)}, \epsilon^{(1)}))^{(N^{(1)}-1)} Q(\epsilon^{(1)}, \epsilon^{(1)})$	$(1-P^{(1)})^{(N^{(1)}-1)} \cdot P^{(1)}/(1-\epsilon_N^{(1)})$

As is seen in our models adopted in this paper $\det Q^{*(i)} = 1$ and $Q^{*(i)} \in SL(2, R)$. In addition it can be expected that in general cases under the conditions a), b) and c), Q defined by $Q = |\det T|^{-1/m} T$ ($\det Q = \pm 1$) produces Q^* satisfying $\det Q^{*(i)} = 1$ and $Q^{*(i)} \in SL(m, R)$. We thus proceed further supposing $\det Q^{*(i)} = 1$ and $Q^{*(i)} \in SL(m, R)$. Hence we can define $G^{(i)}$ as the smallest closed subgroup of $SL(m, R)$ including all kinds of the irreducible transfer matrices of the i -th kind $Q^{*(i)}$.

Then an infinite sequence of $Q^{*(i)} \{Q_l^{*(i)}; l=1, 2, 3, \dots\}$ can be regarded as a sequence of mutually independent $G^{(i)}$ -valued random variables with a common distribution $\mu^{(i)}$. In this stage Furstenberg's convergent theorem is applied to the set of irreducible transfer matrices $G^{(i)}$.

Furstenberg's theorem If

- 1) $G = \{Q^*\}$ is a noncompact subgroup of $SL(m, R)$ such that no subgroup

of finite index is reducible

$$2) \int \|Q^*\| d\mu(Q^*) < \infty,$$

$$(\|Q^*\| = \sup \|Q^* X\|, X \in R^m \text{ and } \|X\| = 1)$$

then

$$P\left(\lim_{n \rightarrow \infty} \frac{1}{n} \log \|(\prod_{i=1}^n Q_i^*) X\| = \gamma > 0\right) = 1$$

for any $X \in (R^m - \{0\})$.

If $G^{(t)}$ satisfies the above conditions for an energy ϵ or $\hbar\omega$, then

$$P\left(\lim_{l' \rightarrow \infty} \frac{1}{l'} \log \|(\prod_{i=1}^{l'} Q_i^{*(t)}) X_0\| = 2\gamma^* > 0\right) = 1 \tag{4.14}$$

on the sample space $\mathcal{Q}^{(t)}$ for all $X_0 \in (R^m - \{0\})$. We thus have

$$P\left(\lim_{l' \rightarrow \infty} \left(\frac{l'}{n^{(t)}(l')}\right) \frac{1}{l'} \log \|(\prod_{i=1}^{l'} Q_i^{*(t)}) X_0\| = P^{(t)} 2\gamma^{*(t)} = 2\gamma^{(t)} > 0\right) = 1 \tag{4.15}$$

on $\mathcal{Q}^{(t)}$ for all $X_0 (\neq 0)$, where $n^{(t)} = n^{(t)}(l')$ is the number of Q contained in the product of the sequence of $Q^{*(t)}$'s, i.e.,

$$\prod_{j=1}^{n^{(t)}(l')} Q_j = \prod_{i=1}^{l'} Q_i^{*(t)}. \tag{4.16}$$

4.3) It has been shown that there exists a positive number $\gamma^{(t)}$ defined in (4.15), if $G^{(t)}$ satisfies conditions 1) and 2). In this case, for a sufficiently small ϵ there exists an integer $N^{(t)}$ such that for $n \geq N^{(t)}$,

$$f^{(t)} e^{2(\gamma^{(t)} - \epsilon)n^{(t)}(l')} < |X_n^{(t)}(l')|^2 < g^{(t)} e^{2(\gamma^{(t)} + \epsilon)n^{(t)}(l')} \tag{4.17}$$

for each $n^{(t)}(l')$ where $\alpha_n^{(t)}(l')(\omega) = \alpha^{(t)}$. Quantities $f^{(t)}$ and $g^{(t)}$ are some positive numbers.

The relation (4.17) provides us, however, with only partial information about the exponential growth of the wave function. It does not guarantee the existence of γ in (3.6) with probability 1 on \mathcal{Q} for any $X_0 \neq 0$. There are three cases.

case a) The existence of $\gamma^{(t)}$ (4.15) is guaranteed for every subsets.
 ($i = 1 \sim s$)
 (for a given energy ϵ or $\hbar\omega$)

case b) The existence of $\gamma^{(t)}$ is guaranteed for at least on subset but at the same time it is not guaranteed for at least one of other subsets.

case c) The existence of $\gamma^{(t)}$ cannot be guaranteed for any subset. It can be shown, in the same way as we have already seen,²²⁾ that at least in case a) the existence of (3.6) is guaranteed for the energy and

$$\gamma = \gamma^{(1)} = \gamma^{(2)} = \dots = \gamma^{(s)}. \tag{4.18}$$

Now we obtain the following Furstenberg-type theorem. If the property (4.14) is proved for every $G^{(i)}$ ($i=1 \sim s$), then we have

$$P\left(\lim_{n \rightarrow \infty} \frac{1}{n} \log \| (\prod_{i=1}^n Q_i) X_0 \| = 2\gamma > 0\right) = 1 \tag{4.19}$$

or

$$P\left(\lim_{n \rightarrow \infty} \frac{1}{n} \log \| (\prod_{i=1}^n T_i) X_0 \| = 2\gamma > 0\right) = 1 \tag{4.19'}$$

on \mathcal{Q} for all $X_0 \in (R^m - \{0\})$. With the Furstenberg-type theorem localization of eigenstates can be discussed in a wide variety of one-dimensional systems.

4.4) As mentioned above, a sufficient condition for the exponential growth of the wave function (3.6) is that the property (4.14) or (4.15) is proved for every subgroup $G^{(i)}$ or subset $\mathcal{Q}^{(i)}$. Conditions 1) and 2) are sufficient conditions for the property (4.14) to be valid.

The condition 2) can be made to be satisfied when we omit from $G^{(i)}$ the irreducible transfer matrices corresponding to the irreducible sequences the lengths of which are larger than a sufficiently large integer $N^{(i)} > N/P^{(i)}$. The finite integer $N^{(i)}$ can be made as large as one hopes so that the integer does not affect physical phenomena, that is, the probability distribution of the subtracted set can be made as small as one hopes by taking a large integer N . It can be done by constructing the set \mathcal{Q} by subtracting from \mathcal{Q}^0 a set $\{\omega_j^0\}$ with a sufficiently small measure $\epsilon_n > 0$. Apparently it is adequate here to understand

$$\begin{aligned} \{\omega_j^0\}^{(i)} \in \mathcal{Q}^{(i)}, \mu^i(\{\omega_j^0\}^{(i)}) &= P^{(i)} \epsilon_N^{(i)}, \quad (i=1 \sim s) \\ \{\omega_j^0\} &= \sum_{i=1}^s \{\omega_j^0\}^{(i)}, \quad \epsilon_N = \sum_{i=1}^s p^{(i)} \epsilon_N^{(i)}, \end{aligned} \tag{4.20}$$

and each $\{\omega_j^0\}^{(i)}$ consists of sample systems each sample of which includes at least one irreducible sequence of the i -th kind, the length of which is greater than $N^{(i)}$. The set \mathcal{Q} should therefore be understood, when it is necessary, as an aggregation of all sample systems in which each sample $\omega \in \mathcal{Q}^{(i)}$ ($i=1 \sim s$) can be represented by two infinite sequences of the irreducible sequences the length of which is less than or equal to $N^{(i)}$, describing the right and the left part of the chain.

After omitting the special samples $\{\omega_j^0\}$ from \mathcal{Q}^0 , we have only to consider the condition 1) for each subset $\mathcal{Q}^{(i)}$. It is now evident that we can make the finite integer N as large as one hopes and accordingly the positive value ϵ_N can be made as small as one hopes so that the existence of ϵ_N does not affect physical phenomena. Evidently probability 1 on \mathcal{Q} means, when it is

necessary, probability $1 - \varepsilon_N$ on Ω^0 .

§ 5. Applications of the theorem

As discussed in the last section a sufficient condition for the property (3.6) is, in the meaning mentioned above, that condition 1) is satisfied for each $G^{(i)}$ ($i=1 \sim s$).

We want to discuss here the condition on our simple models. In these models $m=2$ and excluding some special exceptions, F -condition is a tractable sufficient condition. Now we can tentatively summarize about the property (3.6) on these models using F -condition. We have the following conclusions about the property.

On the models in § 2 F -condition is satisfied for all energies, except for at most several special point energies on each model, irrespective of the detailed characteristics of each model. Most of the exceptions, the special point energies, are believed to appear only because we did not check F -condition extensively on each model. Moreover we can find from a general consideration of the condition that it is quite difficult to find disordered one-dimensional systems which do not satisfy F -condition. This seems to suggest that the property (3.6) is a character almost common in the one-dimensional infinite disordered systems.

§ 6. Conclusions and discussion*)

The first conclusion is that a Furstenberg-type theorem is obtained for the products of random matrices representing a (multi-) Markov-chain. The theorem is applicable to any stationary Markov-process which can be represented by a sequence of matrices belonging to $SL(m, R)$. Moreover the theorem is applicable to some stationary Markov-processes which can be represented by sequences of real matrices all of them do not belong to $SL(m, R)$. On this point only $Q^{*(i)} \in SL(m, R)$ ($i=1 \sim s$) is requested in this paper to establish the theorem. The theorem is thus applicable to a wide variety of random systems.

Our second conclusion is, as a result of the first one, that the property (3.6) is satisfied at least, except for at most several point energies, for all energies on each model. The property (3.6) guarantees the conclusions below using related theorems essentially generalized in the previous paper.**)

The third conclusion is that any infinite chain which belongs to the category considered can be made to have an exponentially localized solution for a given energy $E^{**})$ for which the property (3.6) is satisfied, by modifying

*) An essential part of the conclusions has been reported in Int. Conf. "Physics in One-Dimension" (Fribourg, 1980). (Ref. 23.)

***) E means $\hbar\omega$ (or ω^2) in model 1 and ε in models 2 and 3.

a matrix element to get a suitable value, except for the chains with measure zero on \mathcal{Q} .

The fourth conclusion is that the spectrum is not absolutely continuous, that is, no density of states exists which is Lebesgue integrable.

The fifth conclusion is that almost all of the eigenstates (for the energy E) are exponentially localized, in infinite systems, with probability 1 on \mathcal{Q} , in the sense that the following relation holds with probability 1 on \mathcal{Q} ,

$$|G_{n,m}(E-i0)| < O(\exp\{-\gamma(E)|n-m|\})$$

in the limit $|n-m| \rightarrow \infty$.

Here, in model 1, Eq. (2.1') is used to define $G_{n,m}$ and E means ω^2 .

The sixth conclusion is that the weak absence of diffusion³⁰⁾ takes place also with probability 1 on \mathcal{Q} , for the above-mentioned energies in models 2 and 3.

Concerning the first conclusion one further problem is to discuss any stationary Markov-process which can be represented by a sequence of matrices belonging to $SL(m, C)$. This will complete the approach and we will be able to discuss almost all types of one-dimensional disordered linear systems with short-range interaction.

Other problems are to discuss the rate of exponential growth γ and the feature of spectral densities. Recently we can see a remarkable progress on the later made by mathematicians^{30)~31)} though the model is different from ours. Their model has point spectrum with probability 1 and the eigenfunctions are exponentially localized. The former guarantees the absence of diffusion in Anderson's sense and the later does the vanishing of the DC conductivity in one-dimension.

It is finally concluded that Mott and Twose's conjecture³²⁾ is almost justified in one-dimensional disordered systems.

We have now a new type of problem. How can one-dimensional systems be conductive? Residual interactions in three dimension, interaction between particles^{33), 34)} and interaction with the other degree of freedom³⁵⁾ such as phonons³⁴⁾ are some candidates for the mechanism of the conductivity. The problem is left for the future.

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